Nonlocal effects in the conserved Kardar-Parisi-Zhang equation

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By using the dynamic renormalization group approach, we analyze a nonlocal conserved Kardar-Parisi-Zhang equation with spatially correlated conservative noise in order to study the effect of the long-range nature of interactions coupled with spatially correlated noise on the dynamics of a volume conserving surface. The roughness of the surface depends on both the long-range interaction strength and the spatial correlation parameter. The surface becomes less rough by the long-range interaction, while it becomes more rough by the spatial correlation of noise. We also study the nonlocal conserved Kardar-Parisi-Zhang equation with spatially correlated nonconservative noise.

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For the past decade the kinetic roughening of surfaces has attracted much interest [1]. Various kinetic growth models and related continuum growth equations have been investigated numerically and analytically by measuring the scaling exponents that characterize the asymptotic behavior of the surface roughness on the large length scale and in a long time limit. The most well known continuum equation is the Kardar-Parisi-Zhang (KPZ) equation [2] which has become a paradigm for the kinetic roughening phenomena. Nevertheless, there is poor agreement between the KPZ theory and the experiments [3]. The discrepancy between the KPZ theory and the experiments has spurred considerable theoretical activities involving modifications of the KPZ theory, such as a conserved KPZ(CKPZ) equation [4,5] as well as correlated [6], non-Gaussian [7], and quenched noise [8,9]. However, most of these studies are related to the short-range nature of interaction in nonlinear terms.

Recently, Mukherji and Bhattacharjee proposed a phenomenological equation in the presence of long-range interactions, the nonlocal KPZ(NKPZ) equation which has a nonlinear term with the long-range interaction as coupling the gradients at two different points [10]. The nonlocal conserved KPZ(NCKPZ) equation with the same kind of long-range interaction was also studied [11]. Using the dynamic renormalization-group (RG) approach, it was found that the roughness of the surface changes and several distinct phases appear. Chattopadhyay studied the generalization of the NKPZ equation with spatially correlated noise [12]. The correlated noise coupled with the long-range interactions produces different phases in different regimes from the phases formed by the NKPZ equation with white noise. Here we extend the NCKPZ equation [11] to the case of spatially correlated noise.

The continuum equation for the coarse-grained height variable $h(\mathbf{r},t)$, which describes the surface as a function of coordinate $\mathbf{r}$ and time t, is given by

$$\frac{\partial h(\mathbf{r},t)}{\partial t} = -K\nabla^4 h(\mathbf{r},t) + \eta_c(\mathbf{r},t) - \frac{1}{2}\nabla^2 \int d\mathbf{r}'\,\vartheta(\mathbf{r}') \times \nabla h(\mathbf{r}+\mathbf{r}',t)\cdot\nabla h(\mathbf{r}-\mathbf{r}',t), \tag{1}$$

where the parameter K is a constant. The conservative noise $\eta_c(\mathbf{r},t)$ has a power law correlation of the form

$$\langle \eta_c(\mathbf{r},t)\,\eta_c(\mathbf{r}',t')\rangle \sim \nabla^2 |r-r'|^{2\sigma-d}\,\delta(t-t'), \tag{2}$$

where d is the substrate dimension. Spatial correlation is introduced through the noise spectrum $D(\mathbf{k},\omega)$, defined as

$$\langle \eta_c(\mathbf{k},\omega)\rangle = 0,$$
$$\langle \eta_c(\mathbf{k},\omega)\,\eta_c(\mathbf{k}',\omega')\rangle = 2D(\mathbf{k})\,\delta^d(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega'), \tag{3}$$

where $\eta_c(\mathbf{k},\omega)$ is the Fourier transform of noise $\eta_c(\mathbf{r},t)$. A form of $D(\mathbf{k})$ can be written as $D(\mathbf{k}) = (D_0 + D_\sigma k^{-2\sigma})k^2$, where σ is an exponent characterizing the decay of spatial correlations. The limiting case $\sigma=0$ gives the uncorrelated conservative noise which is a white noise of zero mean with $\langle \eta_c(\mathbf{k},\omega)\,\eta_c(\mathbf{k}',\omega')\rangle = 2D_0\nabla^2\delta^d(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')$. The kernel $\vartheta(\mathbf{r})$ has a short-range part $\lambda_0\delta^d(\mathbf{r})$ and a long-range part $r^{\rho-d}$ in Fourier space $\vartheta(\mathbf{k}) = \lambda_0 + \lambda_\rho k^{-\rho}$. Since the right-hand side of Eq. (1) can be written as the divergence of a current, including the noise term, the total volume under the surface is conserved. This is the feature of conservative noise.

The surface width $W(L,t)$ can be described by the dynamical scaling form $W(L,t) = L^\alpha f(t/L^z)$, where L, α, z, and f are the system size of the substrate, the roughness exponent, the dynamic exponent, and the scaling function, respectively. The scaling function $f(x)$ approaches a constant for $x \gg 1$, and $f(x) \sim x^{\alpha/z}$ for $x \ll 1$. In the absence of nonlinearity ($\lambda_0 = \lambda_\rho = 0$), Eq. (1) becomes a linear equation with spatially correlated noise evolving with the conservative surface diffusion, where the roughness exponent α is $(2-d+2\sigma)/2$ and the dynamic exponent z is four which can be obtained by a dimensional analysis. The linear equation with the spatially correlated conservative noise produces a roughness exponent as a function of noise correlation parameter σ. For $\lambda_\rho = 0$ and $\lambda_0 \neq 0$, Eq. (1) becomes the conserved KPZ equation with spatially correlated conserved noise, which was studied by Family *et al.* [13]. For this local conserved growth equation, the dynamic RG calculation shows that $\alpha = (2-d+2\sigma)/3$ and $z = (10+d-2\sigma)/3$ for $d < 2+2\sigma$. In the limiting case of $\sigma = 0$ and $\lambda_\rho = 0$, Eq. (1) corresponds to that of Sun, Guo, and Grant [4] which gives $\alpha = (2-d)/3$ and $z = (10+d)/3$. In this paper, we study the effects of the long-range interaction of nonlinearity (λ_ρ

$\neq 0$) and spatially correlated noise on the dynamics of a volume-conserving interface. We show that the spatially correlated noise in the presence of the long-range interaction of nonlinearity produces new fixed points and the exponents depend on both parameter ρ of the long-range interaction and the spatial correlation parameter σ of noise. The long-range interaction of nonlinearity makes the surface less rough than that of the short-range interaction, while the spatial correlation of noise makes it more rough.

Under the change of scale, the parameters in Eq. (1) make the changes $K \to b^{z-4}K$, $\lambda_0 \to b^{z+\alpha-4}\lambda_0$, and $\lambda_\rho \to b^{z+\alpha-4+\rho}\lambda_\rho$. The noise strength changes by $D_0 \to b^{z-2\alpha-d-2}D_0$ and $D_\sigma \to b^{z-2\alpha-d-2+2\sigma}D_\sigma$. Thus, when $\sigma=0$, the critical dimensions are $d_c=2+2\rho(\rho>0)$ and $d_c=2(\rho<0)$. If $\sigma>0$, the critical dimensions are given by $d_c=2+2\rho+2\sigma(\rho>0)$ and $d_c=2+2\sigma(\rho<0)$ for any λ_ρ. Thus, when $\rho>0$ and $\sigma>0$, if $d<d_c=2+2\rho+2\sigma$, a new fixed point depending on ρ and σ is expected. If $d \geqslant 2+2\rho+2\sigma$, the nonlocality of the nonlinear term and the correlated noise become irrelevant and the surface is controlled by the linear equation.

Following the dynamic RG procedure [6], integrating out fast modes in the momentum shell $e^{-\ell}\Lambda \leqslant |\mathbf{k}| \leqslant \Lambda$ and performing the rescalings $r \to br$, $t \to b^z t$, and $h \to b^\alpha h$, we derive the following flow equations for the coefficients, in a one-loop approximation:

$$\frac{dK}{d\ell}=K\left\{z-4-\frac{B_dD(1)}{K^3}\vartheta(2)\vartheta(1)\times\frac{d-4+3f(1)+g(1)}{4d}\right\}, \tag{4}$$

$$\frac{dD(k)}{d\ell}=D(k)[z-2\alpha-d-2-g(k)], \tag{5}$$

$$\frac{d\lambda_x}{d\ell}=\lambda_x(z+\alpha-4+x), \tag{6}$$

where $x=0$ or ρ, respectively. Here $f(q)=\partial \ln \vartheta(k)/\partial k|_{k=q}$ and $g(q)=\partial \ln D(k)/\partial k|_{k=q}$. Since the diagrams contributing to $D(k)$ have prefactors proportional to k^4, they correspond to higher derivatives in the original noise spectrum. Note that two scaling relations $z+\alpha=4$ and $z+\alpha=4-\rho$, which result from the nonrenormalization of λ_0 and λ_ρ in Eq. (6), are the results of a one-loop approximation [14].

To obtain the RG recursion relations, we define the dimensionless effective coupling constant $U_{xy}^2=B_d(\lambda_x^2 D_y)/K^3$, where $x=0$ or ρ and $y=0$ or σ. Here $B_d=S_d/(2\pi)^d$, S_d being the surface area of a d-dimensional unit sphere. By using these dimensionless effective coupling constants, we find that the RG recursion relations are as follows:

$$\frac{dU_{00}}{d\ell}=\frac{U_{00}}{2}[2-d+3(\cdot)], \tag{7}$$

$$\frac{dU_{0\sigma}}{d\ell}=\frac{U_{0\sigma}}{2}[2-d+2\sigma+3(\cdot)], \tag{8}$$

$$\frac{dU_{\rho 0}}{d\ell}=\frac{U_{\rho 0}}{2}[2-d+2\rho+3(\cdot)], \tag{9}$$

$$\frac{dU_{\rho\sigma}}{d\ell}=\frac{U_{\rho\sigma}}{2}[2-d+2\rho+2\sigma+3(\cdot)], \tag{10}$$

where the centered dot is given by

$$(\cdot)=\frac{1}{4d}\{(d-4)[(U_{00}^2+U_{0\sigma}^2)+(1+2^{-\rho})(U_{00}U_{\rho 0}+U_{0\sigma}U_{\rho\sigma})+2^{-\rho}(U_{\rho 0}^2+U_{\rho\sigma}^2)]-3\rho[(U_{00}U_{\rho 0}+U_{0\sigma}U_{\rho\sigma})+2^{-\rho}(U_{\rho 0}^2+U_{\rho\sigma}^2)]-2\sigma[U_{0\sigma}^2+(1+2^{-\rho})U_{0\sigma}U_{\rho\sigma}+2^{-\rho}U_{\rho\sigma}^2]\}. \tag{11}$$

Now let us consider the following four sets of effective coupling constants: $(U_{00},U_{\rho 0})$, $(U_{0\sigma},U_{\rho\sigma})$, $(U_{00},U_{0\sigma})$, and $(U_{\rho 0},U_{\rho\sigma})$. From the RG recursion relations, we find axial fixed points for the four sets of effective coupling constants. Unlike the case of KPZ and NKPZ equation with correlated noise [6,12], there is no off-axis fixed point. It is because the equation for the ratio of effective coupling constants in each space rules out the existence of any off-axis fixed point [e.g., $d(U_{00}/U_{\rho 0})/d\ell=-\rho(U_{00}/U_{\rho 0})$].

In $(U_{00},U_{\rho 0})$ plane, there exist two axial fixed points. When $\rho=0$, the fixed point, $U_{00}^{*2}=4d(2-d)/3(4-d)$ which is obtained by setting $dU_{00}/d\ell=0$, is stable for $d<2$. Using Eqs. (4),(7), and the relation $z+\alpha=4$, the roughness and the dynamic exponents are obtained as

$$\alpha=(2-d)/3, \quad z=(10+d)/3. \tag{12}$$

At physical dimension $d=2$, the surface width is logarithmically rough. These results at this fixed point are in agreement with those of Sun, Guo, and Grant [4]. When $\rho>0$, the effective nonlinearity $U_{\rho 0}$ is dominant over U_{00}. Thus, the phase in space (U_{00}, $U_{\rho 0}$), except for $U_{\rho 0}=0$, is determined by the long-range λ_ρ term in Eq. (1). By setting $dU_{\rho 0}/d\ell=0$, the fixed point $U_{\rho 0}^{*2}=4d(2-d+2\rho)/3(4-d+3\rho)2^{-\rho}$ is obtained. This fixed point is stable for $d<2+2\rho$ and the exponents are given by

$$\alpha=(2-d-\rho)/3, \quad z=(10+d-2\rho)/3. \tag{13}$$

These exponents are determined by Eqs. (4) and (9) with $z+\alpha=4-\rho$ and both exponents decrease by the parameter ρ of the long-range interaction. If $d<2-\rho$, the surface is the rough phase with a positive roughness exponent, while if $d>2-\rho$, it is the smooth phase with a negative value of the roughness exponent. In particular, at $d=2$ the nonzero $U_{\rho 0}$ term with $\rho>0$ can make the surface less rough than the logarithmically rough phase of the case $\rho=0$. In contrast, for $\rho<0$, the fixed point $U_{\rho 0}^{*2}$ is irrelevant on the grounds that U_{00} is dominant over $U_{\rho 0}$. $U_{\rho 0}^{*2}$ is stable for $d<2+2\rho$, only if $U_{00}=0$. The results in this space $(U_{00},U_{\rho 0})$ are the same of those of Ref. [11].

In plane $(U_{0\sigma},U_{\rho\sigma})$, when $\rho=0$, the surface is governed by the fixed point, $U_{0\sigma}^{*2}=4d(2-d+2\sigma)/3(4-d+2\sigma)$, if $d<2+2\sigma$ and $\sigma>0$. At this fixed point, the exponents are given by

$$\alpha = (2-d+2\sigma)/3, \quad z = (d+10-2\sigma)/3, \tag{14}$$

from Eqs. (4) and (8) with the relation $z+\alpha=4$. Unlike the case of Eq. (13), Eq. (14) shows that due to the presence of spatial correlation in noise the roughness exponent increases: that is, the surface becomes more rough than that in the case of white noise. This is different from the role of ρ which makes the surface less rough. When $\rho>0$, the fixed point $U_{0\sigma}^{*2}$ crosses over into the fixed point $U_{\rho\sigma}^{*2}=4d(2-d+2\rho+2\sigma)/3(4-d+3\rho+2\sigma)2^{-\rho}$, which is stable for $d<2+2\rho+2\sigma$. At this fixed point, the dynamic and the roughness exponents are obtained from Eqs. (4) and (10) with $z+\alpha=4-\rho$ as

$$\alpha = (2-d-\rho+2\sigma)/3, \quad z = (10+d-2\rho-2\sigma)/3. \tag{15}$$

When $d<2-\rho+2\sigma$, the surface is rough with a positive roughness exponent, while if $d>2-\rho+2\sigma$, it is smooth with a negative one. This is the same as that of Eq. (14) if $d\to d+2\sigma$. When $\rho<0$, the fixed point $U_{\rho\sigma}^{*2}$ becomes irrelevant and the fixed point $U_{0\sigma}^{*2}$ is relevant, except for $\lambda_0=0$. Note that in Eqs. (13), (14), and (15) both the correlated noise and the long-range interaction of the CKPZ nonlinearity decrease the value of the dynamical exponent.

In plane $(U_{00},U_{0\sigma})$, the behavior of the RG flows is the same as the one for the CKPZ equation with spatially correlated noise studied by Family *et al.* [13]. The fixed point U_{00}^{*2} is stable only at $\sigma=0$. For $\sigma>0$, it crosses over into the correlated fixed point $U_{0\sigma}^{*2}$, which is stable for $d<2+2\sigma$. At this correlated fixed point, the exponents are given by Eq. (14) which is valid if $\sigma>0$. This is different from the case of correlated noise in the KPZ equation where a transition from uncorrelated to correlated behavior occurs at a finite σ_c.

In plane $(U_{\rho 0},U_{\rho\sigma})$, for $\sigma=0$ the surface is governed by the fixed point $U_{\rho 0}^{*2}$. If $\sigma>0$, the fixed point $U_{\rho 0}^{*2}$ becomes unstable and it crosses over into the fixed point $U_{\rho\sigma}^{*2}$. This fixed point is stable for $d<d+2\rho+2\sigma$, even though $\rho<0$. At this fixed point the exponent is given by Eq. (15).

We have also studied Eq. (1) with spatially correlated nonconservative noise instead of a spatially correlated conservative one. In the NCKPZ equation with spatially correlated nonconservative noise, the average height is not conserved [13]. In the spatially correlated nonconservative noise, form $D(\mathbf{k})$ in Eq. (3) can be written as $D(\mathbf{k})=D_0+D_\sigma k^{-2\sigma}$. There are four sets of axial fixed points and the results of the spatially correlated nonconservative case are the same as those of the spatially correlated conservative case if the dimensionality is replaced by $d\to d-2$ in Eqs. (7)–(10) and symbol is replaced by

$$\begin{aligned}(\cdot) = \frac{1}{4d}\{ &(d-6)[(U_{00}^2+U_{0\sigma}^2)+(1+2^{-\rho})(U_{00}U_{\rho 0}\\ &+U_{0\sigma}U_{\rho\sigma})+2^{-\rho}(U_{\rho 0}^2+U_{\rho\sigma}^2)]-3\rho[(U_{00}U_{\rho 0}\\ &+U_{0\sigma}U_{\rho\sigma})+2^{-\rho}(U_{\rho 0}^2+U_{\rho\sigma}^2)]-2\sigma[U_{0\sigma}^2\\ &+(1+2^{-\rho})U_{0\sigma}U_{\rho\sigma}+2^{-\rho}U_{\rho\sigma}^2]\}. \end{aligned} \tag{16}$$

As the case of the spatially correlated conservative noise, for any finite ρ and σ, the correlated fixed points are stable if $d<d_c$.

In conclusion, we have studied the nonlocal conserved KPZ equation with spatially correlated conservative noise and nonconservative noise by using the dynamic RG approach. In both cases, the long-ranged nature of interactions coupled with spatially correlated noise produces new fixed points at which we obtained the roughness and the dynamic exponents as a function of the parameter of long-range interactions ρ, spatial correlation parameter σ, and the dimensionality d. The nonlocal λ_ρ term with positive ρ makes the surface less rough than in the case of $\lambda_\rho=0$, while the spatial correlation of noise makes it more rough.

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